Photon-added Barut–Girardello coherent states of the pseudoharmonic oscillator

2002 J. Phys. A: Math. Gen. 35 7205

(http://iopscience.iop.org/0305-4470/35/33/315)

J. Phys. A: Math. Gen. **35** (2002) 7205–7223 PII: S0305-4470(02)34790-5

Photon-added Barut–Girardello coherent states of the pseudoharmonic oscillator

Dušan Popov

Department of Physics, University 'Politehnica' of Timişoara, Piaţa Regina Maria No 1, Of. Postal 5, 1900 Timişoara, Romania

E-mail: dpopov@etv.utt.ro

Received 12 March 2002
Published 7 August 2002
Online at stacks.iop.org/JPhysA/35/7205

Abstract

In a previous paper (Popov D 2001 *J. Phys. A: Math. Gen.* **34** 5283) we have constructed the coherent states (CSs) of the pseudoharmonic oscillator (PHO). These states are of the Barut–Girardello type. In the present paper, we have constructed and investigated some properties of the photon-added Barut–Girardello coherent states (PA-BGCSs) of the PHO. These states are not orthogonal but are normalized and satisfy a unity resolution relation (the completeness relation). We have found the analytical form for the positive weight function in the resolution of unity. By using these states, we have calculated some expectation values (the first two powers of the number operator N), which have evinced some non-classical properties. In order to examine the statistical properties of a PHO gas, which obeys the quantum canonical distribution, the diagonal P-representation of the density operator $\rho^{(m)}$ is constructed and the thermal expectation values are calculated. All these quantities are expressed in terms of Meijer G-functions, and so the PA-BGCSs are a new field of application for these functions. Also, the time dependence of the PA-BGCSs is examined. If in the obtained results we put $m = 0$, we recover all the results from our previous paper.

PACS numbers: 03.65.C, 03.65.−w, 33.10.C, 42.50.A

1. Introduction

An anharmonic potential, suitable for the treatment of molecular vibrations, which permits an exact mathematical approach is the pseudoharmonic oscillator (PHO) potential [1–9]. The effective potential of the PHO is

$$V_J(r) = \frac{m_r\omega^2}{8} r_0^2 \left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 + \frac{\hbar^2}{2m_r} J(J+1)\frac{1}{r^2} \tag{1}$$

where m_r is the reduced mass, ω is the angular frequency, r_0 is the equilibrium distance between the nuclei of the diatomic molecule and $J = 0, 1, 2, \ldots$ is the rotational quantum number.

The radial eigenfunctions and eigenvalues have been calculated in [2]:

$$u_v^\alpha(r) \equiv r R_{vJ}(r) = \frac{1}{B}\left[\frac{B^3 v!}{2^\alpha \Gamma(v+\alpha+1)}\right]^{\frac{1}{2}} (Br)^{\alpha+\frac{1}{2}} \exp\left(-\frac{B^2}{4}r^2\right) L_v^\alpha\left(\frac{B^2}{2}r^2\right) \tag{2}$$

$$E_{vJ} = \hbar\omega_0 (2v+1) + \hbar\omega_0\alpha - m_r\omega_0^2 r_0^2. \tag{3}$$

Here we have used the notations

$$\omega = 2\omega_0 \qquad B = \left(\frac{m_r\omega_0}{\hbar}\right)^{\frac{1}{2}} \qquad \alpha = \left[\left(J+\frac{1}{2}\right)^2 + \left(Br_0^2\right)^2\right]^{\frac{1}{2}} \tag{4}$$

where $\Gamma(x)$ is Euler's gamma function, $L_v^\alpha(x)$ is the generalized Laguerre polynomial and ω_0 is the angular frequency of the usual three-dimensional harmonic oscillator (HO-3D). At the harmonic limit ($r_0 \to 0$ and $\omega \to 2\omega_0$), the PHO potential leads to the usual HO-3D potential $V_J(r) = \frac{1}{2}m_r\omega_0^2 r^2 +$ centrifugal term.

In the following we recall the main results of our previous paper [9] (which we call paper PI) that are useful for the present paper. Passing to the dimensionless variable $y = \left(\frac{m_r\omega_0}{\hbar}\right)^{\frac{1}{2}} r = Br$ allows us to rewrite the radial equation as follows

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{2}y^2 + \frac{1}{2}\left(\alpha^2 - \frac{1}{4}\right)\frac{1}{y^2} - (2v+\alpha+1)\right] u_v^\alpha(y) = 0 \tag{5}$$

where the dimensionless reduced Hamiltonian $H_\alpha^{(\mathrm{red})}$ of the PHO appears

$$H_\alpha^{(\mathrm{red})}(y) = \frac{1}{\hbar\omega_0}\left[H_\alpha(y) + m_r\omega_0^2 r_0^2\right] \equiv -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{2}y^2 + \frac{1}{2}\left(\alpha^2 - \frac{1}{4}\right)\frac{1}{y^2}. \tag{6}$$

We have to find the $SU(1,1)$ group operators [9]

$$K_3 = \tfrac{1}{2}H_\alpha^{(\mathrm{red})}(y) \tag{7}$$

$$K_\pm = K_1 \pm \mathrm{i}K_2 = \frac{1}{2}\left(\pm y\frac{\mathrm{d}}{\mathrm{d}y} \pm \frac{1}{2} - y^2 + 2v + \alpha + 1\right) \tag{8}$$

which satisfy the following commutation relations

$$[K_3, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_3. \tag{9}$$

The Casimir operator K^2 for any irreducible representation is the identity times a number

$$K^2 = K_3^2 - K_1^2 - K_2^2 = K_3^2 + K_3 - K_-K_+ = k(k-1). \tag{10}$$

So, an irreducible representation of $SU(1,1)$ is determined by a single real number k (called the Bargmann index).

Here we are only interested in the unitary irreducible representations known as positive discrete series, where $k > 0$. The corresponding state is spanned by the complete orthonormal basis of the number state $|v, k\rangle$ (where $v = 0, 1, 2, \ldots, \infty$ is the vibrational quantum number) of the PHO Hilbert space:

$$\langle v, k|v', k\rangle = \delta_{vv'} \qquad \sum_{v=0}^{\infty} |v, k\rangle\langle v, k| = 1. \tag{11}$$

The number state vectors $|v, k\rangle$ and the reduced radial eigenfunctions $u_v^{\alpha}(y)$ satisfy the same eigenequations with respect to the raising and lowering operators $K_{\pm}$

$$K_{+}|v, k\rangle = \sqrt{(v+1)(v+2k)}|v+1, k\rangle \tag{12}$$

$$K_{-}|v, k\rangle = \sqrt{v(v+2k-1)}|v-1, k\rangle \tag{13}$$

if we make the following identification

$$\alpha = 2k - 1 \tag{14}$$

i.e. the rotational parameter α plays the role of the Bargmann index. Further on, we use the k index instead of the α index.

The Barut–Girardello coherent states (BGCSs) $|z, k; 0\rangle \equiv |z, k\rangle$ can be constructed as eigenstates of the lowering generator K_{-} [10]

$$K_{-}|z, k\rangle = z|z, k\rangle \tag{15}$$

where z is an arbitrary complex number.

One can represent the eigenstates $|z, k\rangle$ as the superposition of the complete orthonormal basis $|v, k\rangle$ of the PHO Hilbert space

$$|z, k\rangle = C_0(|z|)\sum_{v=0}^{\infty}\frac{z^v}{\sqrt{v!\Gamma(v+2k)}}|v, k\rangle \tag{16}$$

where we have used the following notation for the normalization constant

$$C_0(|z|) = \sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}}. \tag{17}$$

The purpose of this paper is to construct an important special family of states obtained by adding photons to a conventional BGCS of the PHO $|z, k\rangle$. We call these states the photon-added Barut–Girardello coherent states (PA-BGCSs) of the PHO.

It should be recalled here that, for the case of the usual harmonic oscillator (HO), the photon-added coherent states (PACSs) were introduced for the first time by Agarwal and Tara [11]. Some applications and properties of these states have been the object of extensive studies recently [12, 13].

The idea of adding photons to the variously-defined coherent states, initiated by Agarwal and Tara [11], can of course be applied to any type of coherent state.

2. Construction of the PA-BGCS of the PHO

PACSs of the PHO $|z, k; m\rangle$ can be obtained by repeated application of the raising operator K_{+} (8) to the conventional BGCSs of the PHO $|z, k\rangle$ (16)

$$|z, k; m\rangle = N_m(|z|)(K_{+})^m|z, k\rangle \tag{18}$$

where m is a positive integer being the number of added quanta (or added photons), and $N_m(|z|)$ is the normalization constant.

Because the commutation relations (9) are not canonical commutation rules of the Heisenberg type (i.e. rules characteristic of bosons, or particularly photons) for one mode or multi-modes, in order to avoid any misunderstanding we must note that, in this paper, the word ‘photon’ in the expression ‘photon-added coherent states’ is used in a non-physical sense. Maybe, instead of ‘added photons’ it would be better to say ‘added quanta’.

Using equations (16) and (8), we obtain

$$(K_+)^m|z,k\rangle = C_0(|z|)\sum_{v=0}^{\infty}\frac{z^v}{\sqrt{\rho_m(v;k)}}|v+m;k\rangle \tag{19}$$

where we have used the notation

$$\rho_m(v;k) \equiv \frac{[\Gamma(v+1)]^2[\Gamma(v+2k)]^2}{\Gamma(v+m+1)\Gamma(v+m+2k)}. \tag{20}$$

So, the PA-BGCS becomes

$$|z,k;m\rangle = C_m(|z|)\sum_{v=0}^{\infty}\frac{z^v}{\sqrt{\rho_m(v;k)}}|v+m;k\rangle \tag{21}$$

where

$$C_m(|z|) \equiv N_m(|z|)C_0(|z|) \tag{22}$$

is a new normalization constant which is calculated below.

2.1. Non-orthogonality

The PA-BGCSs of the PHO must be normalized but not orthogonal

$$\langle\sigma,k;n|z,k;m\rangle = C_n(|\sigma|)C_m(|z|)\sum_{v',v=0}^{\infty}\frac{(\sigma^*)^{v'}z^v}{\sqrt{\rho_n(v';k)\rho_m(v;k)}}\langle v'+n;k|v+m;k\rangle. \tag{23}$$

Due to the orthogonality relation of the number vectors (11), it follows that

$$\begin{aligned}\langle\sigma,k;n|z,k;m\rangle &= C_n(|\sigma|)C_m(|z|)(\sigma^*)^{m-n}\frac{\Gamma(m+1)\Gamma(m+2k)}{\Gamma(m-n+1)\Gamma(m-n+2k)\Gamma(2k)}\\ &\quad\times {}_2F_3(m+1,m+2k;m-n+1,m-n+2k,2k;\sigma^*z)\end{aligned} \tag{24}$$

where m and n are positive integers, $m \geqslant n$ and ${}_2F_3(\ldots;\sigma^*z)$ is the generalized hypergeometric series (see appendix A) [14].

One way to obtain a more explicit form of the overlap product (23) is to relate it to Meijer's G-function (see appendix A) [14, 15]

$$\begin{aligned}\langle\sigma,k;n|z,k;m\rangle &= C_n(|\sigma|)C_m(|z|)(\sigma^*)^{m-n}\\ &\quad\times G_{2,4}^{1,2}\left(-\sigma^*z\,\middle|\,\begin{matrix}-m, & 1-m-2k & & \\ 0, & n-m, & 1+n-m-2k, & 1-2k\end{matrix}\right).\end{aligned} \tag{25}$$

If $n \neq m$ and $\sigma \neq z$, the PA-BGCSs are not orthogonal.

2.2. Normalization

By equalizing the above quantities, we obtain the normalization

$$\langle z,k;m|z,k;m\rangle = 1 \tag{26}$$

from which results the normalization constant $C_m(|z|)$

$$C_m(|z|) = \left[G_{2,4}^{1,2}\left(-|z|^2\,\middle|\,\begin{matrix}-m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k\end{matrix}\right)\right]^{-\frac{1}{2}}. \tag{27}$$

Then the normalized PA-BGCSs are

$$|z, k; m\rangle = \left[G^{1,2}_{2,4}\left(-|z|^2 \middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k \end{matrix} \right) \right]^{-\frac{1}{2}}$$
$$\times \sum_{v=0}^{\infty} \frac{z^v}{\Gamma(v+1)\Gamma(v+2k)} \sqrt{\Gamma(v+m+1)\Gamma(v+m+2k)} |v+m; k\rangle \tag{28}$$

where we have used equations (20), (21) and (27).

But, for simplicity of writing, we always write the PA-BGCSs in the manner of equation (21), so this does not create any confusion.

We note here that, after straightforward calculations, the normalized PA-BGCSs can be written also in the following ways:

$$|z, k; m\rangle = \frac{1}{\sqrt{\langle z, k|(K_-)^m (K_+)^m |z, k\rangle}} (K_+)^m |z, k\rangle \tag{29}$$

$$|z, k; m\rangle = [\Gamma(m+1)\Gamma(m+2k)]^{\frac{1}{2}} C_m(|z|) (\sqrt{zK_+})^{-(2k-1)} I_{2k-1}(2\sqrt{zK_+}) |0, k\rangle \tag{30}$$

where $I_\nu(x)$ is the modified Bessel function.

So, for fixed z (and k), the PA-BGCSs $|z, k; m\rangle$ are normalized for any finite natural number m.

2.3. Resolution of unity (or completeness)

From equation (21) we see that the state $|z, k; m\rangle$, as well as the PACSs of the HO [11–13], is a linear combination of all number states starting with $v = m$. In other words, the first m number states $v = 0, 1, \ldots, m-1$, are absent from the state $|z, k; m\rangle$. Then, the unity operator in this space is to be written as [13]

$$\sum_{v=m}^{\infty} |v; k\rangle\langle v; k| = \sum_{v=0}^{\infty} |v+m; k\rangle\langle v+m; k| = I_k^{(m)}. \tag{31}$$

This leads to the following resolution of unity

$$\int\!\!\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} |z, k; m\rangle W_k^{(m)}(|z|) \langle z, k; m| = I_k^{(m)} = \sum_{v=0}^{\infty} |v+m; k\rangle\langle v+m; k|. \tag{32}$$

It is evident that, in the right-hand side of the previous equation, the identity operator on the full Hilbert space does not appear, since the initial m states of the basis set vectors are omitted. Here $I_k^{(m)}$ is only required to be a bounded, positive operator with a densely defined inverse [33].

In order to determine the unknown positive weight function $W_k^{(m)}(|z|)$, we must remember that, at the limit $m = 0$, this weight function must lead to the weight function of the ordinary BGCSs of the PHO (see equation (46) of our previous paper [9])

$$W_k^{(0)}(|z|) = 2K_{2k-1}(2|z|) I_{2k-1}(2|z|) \tag{33}$$

where $K_\nu(x)$ is the νth-order modified Bessel function of the second kind.

Here and below, all the integrals are performed over the whole complex z plane, where

$$z = r \exp \mathrm{i}\varphi \qquad r \in [0, \infty) \qquad \varphi \in [0, 2\pi] \qquad \mathrm{d}^2 z = \mathrm{d}(\mathrm{Re}\, z)\, \mathrm{d}(\mathrm{Im}\, z) = \mathrm{d}\varphi\, r\, \mathrm{d}r. \tag{34}$$

By substituting equation (21) into equation (32) we obtain

$$\int\int_{\mathcal{C}} \frac{d^2 z}{\pi} W_k^{(m)}(|z|)[C_m(|z|)]^2 \sum_{v,v'=0}^{\infty} \frac{(z^*)^{v'} z^v}{\sqrt{\rho_m(v;k)\rho_m(v';k)}} |v+m;k\rangle\langle v+m;k| = I_k^{(m)}. \tag{35}$$

Then, it is obvious that the weight function $W_k^{(m)}(|z|)$ must have the following structure

$$W_k^{(m)}(|z|) = \frac{1}{[C_m(|z|)]^2} |z|^{2m} g_k^{(m)}(|z|). \tag{36}$$

After performing the angular integration, i.e.

$$\int_0^{2\pi} \frac{d\varphi}{\pi} e^{i(v-v')\varphi} = 2\delta_{vv'} \tag{37}$$

equation (35) becomes

$$2\sum_{v=0}^{\infty} \left[\frac{1}{\rho_m(s-m-1;k)} \int_0^{\infty} d\,r r^{2v+2m+1} g_k^{(m)}(r^2)\right] |v+m;k\rangle\langle v+m;k| = I_k^{(m)}. \tag{38}$$

When we perform the variable change $r^2 = x$ and extend the natural values of $v+m$ to complex s such that $v+m \to s-1$, the integral from the above equation is called the Mellin transform [14, 16]

$$\int_0^{\infty} dx\, x^{s-1} g_k^{(m)}(x) = \rho_m(s-m-1;k) = \frac{[\Gamma(s-m)]^2[\Gamma(s-m+2k-1)]^2}{\Gamma(s)\Gamma(s+2k-1)}. \tag{39}$$

Using the definition of Meijer's G-function and the Mellin inversion theorem it follows that [14]

$$\int_0^{\infty} dx\, x^{s-1} G_{p,q}^{m,n}\left(\alpha x \left| \begin{matrix} a_1, & \ldots, & a_n, & a_{n+1}, & \ldots, & a_p \\ b_1, & \ldots, & b_m, & b_{m+1}, & \ldots, & b_q \end{matrix}\right.\right)$$
$$= \frac{1}{\alpha^s} \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}. \tag{40}$$

Comparing equations (39) and (40), we obtain that

$$g_k^{(m)}(x) = G_{2,4}^{4,0}\left(x \left| \begin{matrix} 0, & 2k-1 & & \\ -m, & -m, & 2k-1-m, & 2k-1-m \end{matrix}\right.\right). \tag{41}$$

Then, the weight function becomes

$$\begin{aligned} W_k^{(m)}(|z|) &= \frac{1}{[C_m(|z|)]^2} G_{2,4}^{4,0}\left(|z|^2 \left| \begin{matrix} m, & 2k-1+m & & \\ 0, & 0, & 2k-1, & 2k-1 \end{matrix}\right.\right) \\ &= G_{2,4}^{1,2}\left(-|z|^2 \left| \begin{matrix} -m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k \end{matrix}\right.\right) G_{2,4}^{4,0} \\ &\quad\times \left(|z|^2 \left| \begin{matrix} m, & 2k-1+m & & \\ 0, & 0, & 2k-1, & 2k-1 \end{matrix}\right.\right) \end{aligned} \tag{42}$$

where we have used equation (27) and the multiplication formula for Meijer's G-function (see appendix A) [14]. Because the measure in equation (32) must be necessary positive, the function $W_k^{(m)}(|z|)$ must be also a positive function [13, 16].

Finally, the resolution of unity can be written in an exhaustive manner

$$\int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} \frac{1}{[C_m(|z|)]^2} G_{2,4}^{4,0}\left(|z|^2 \middle| \begin{matrix} m, & 2k-1+m & & \\ 0, & 0, & 2k-1, & 2k-1 \end{matrix}\right) |z,k;m\rangle\langle z,k;m| \\ = I_k^{(m)}. \tag{43}$$

This equation is also called the completeness relation for the PA-BGCSs.

3. Expectation values

The physical utility of the PA-BGCSs in different applications consists in the calculations of the expectation (mean) values of a physical observable A, which characterizes the PHO, and the powers of the particle number operator N, with respect to the PA-BGCSs $|z,k;m\rangle$:

$$\langle z,k;m|A|z,k;m\rangle \equiv \langle A\rangle_{z,k}^{(m)} \\ = [C_m(|z|)]^2 \sum_{v,v'=0}^{\infty} \frac{(z^*)^{v'} z^v}{\sqrt{\rho_m(v';k)\rho_m(v;k)}} \langle v'+m;k|A|v+m;k\rangle. \tag{44}$$

In order to calculate different expectation values (especially those related to the powers of the particle number operator N), it is useful to evaluate the sum $S_n^{(m)}$, with $n = 0, 1, \ldots$ (see appendix B)

$$S_n^{(m)} \equiv \sum_{v=0}^{\infty} \frac{x^v}{\rho_m(v;k)} v^n \\ = \sum_{l=0}^{n} (-1)^l c_l^{(n)} G_{2,4}^{1,2}\left(-x \middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & l, & 1-2k, & 1-2k \end{matrix}\right) \tag{45}$$

where $x = |z|^2$ and the numerical coefficients $c_l^{(n)}$ are easily obtainable from the following operatorial identity

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n = \sum_{l=1}^{n} c_l^{(n)} x^l \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l. \tag{46}$$

In order to simplify the equations, we use the following short notation

$$G_{2,4}^{1,2}\left(-|z|^2 \middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & l, & 1-2k, & 1-2k \end{matrix}\right) \equiv G_{2,4}^{1,2}\left(-|z|^2 \middle| l\right). \tag{47}$$

The expectation values are useful to calculate the variance of a physical observable A in the PA-BGCS $|z,k;m\rangle$:

$$(\sigma_A)_{z,k}^{(m)} \equiv \langle A^2\rangle_{z,k}^{(m)} - \left(\langle A\rangle_{z,k}^{(m)}\right)^2. \tag{48}$$

As an example, we calculate the photon number distribution. It is well known that the number operator N is defined as the operator which diagonalizes the basis for the number states. Using equation (11), we obtain

$$\langle v'+m;k|N|v+m;k\rangle = (v+m)\delta_{vv'} \tag{49}$$

$$\langle v'+m;k|N^2|v+m;k\rangle = (v+m)^2\delta_{vv'}. \tag{50}$$

Then, according to equations (44) and (45), the expectation values for the number operator and its second power are

$$\langle N\rangle_{z,k}^{(m)} = [C_m(|z|)]^2 \left[S_1^{(m)} + mS_0^{(m)}\right] = m - \frac{G_{2,4}^{1,2}(-|z|^2\,|\,1)}{G_{2,4}^{1,2}(-|z|^2\,|\,0)} \tag{51}$$

$$\begin{aligned}\langle N^2\rangle_{z,k}^{(m)} &= [C_m(|z|)]^2 \left[S_2^{(m)} + 2mS_1^{(m)} + m^2 S_0^{(m)}\right] \\ &= m^2 - (2m+1)\frac{G_{2,4}^{1,2}(-|z|^2\,|\,1)}{G_{2,4}^{1,2}(-|z|^2\,|\,0)} + \frac{G_{2,4}^{1,2}(-|z|^2|2)}{G_{2,4}^{1,2}(-|z|^2|0)}.\end{aligned} \tag{52}$$

The second-order correlation function or the intensity correlation function, defined as in [17, 18], is

$$(g^2)_{z,k}^{(m)} = \frac{\langle N^2\rangle_{z,k}^{(m)} - \langle N\rangle_{z,k}^{(m)}}{\left(\langle N\rangle_{z,k}^{(m)}\right)^2}. \tag{53}$$

Equivalently, further information about the inherent statistical properties of the PA-BGCS $|z, k; m\rangle$ follows also from calculating the Mandel parameter $Q_{z,k}^{(m)}$ [19, 20]

$$Q_{z,k}^{(m)} = \frac{(\sigma_N)_{z,k}^{(m)}}{\langle N\rangle_{z,k}^{(m)}} - 1 \tag{54}$$

so that

$$Q_{z,k}^{(m)} = \langle N\rangle_{z,k}^{(m)} \left[(g^2)_{z,k}^{(m)} - 1\right]. \tag{55}$$

A state for which $Q_{z,k}^{(m)} > 0$ (or $(g^2)_{z,k}^{(m)} > 1$) is called super-Poissonian (bunching effect), if $Q_{z,k}^{(m)} = 0$ (or $(g^2)_{z,k}^{(m)} = 1$) the state is called Poissonian, while a state for which $Q_{z,k}^{(m)} < 0$ (or $(g^2)_{z,k}^{(m)} < 1$) is called sub-Poissonian (antibunching effect).

For the PA-BGCSs, the intensity correlation function is

$$(g^2)_{z,k}^{(m)} = \frac{m^2 - m - 2m\frac{G_{2,4}^{1,2}(-|z|^2\,|\,1)}{G_{2,4}^{1,2}(-|z|^2\,|\,0)} + \frac{G_{2,4}^{1,2}(-|z|^2\,|\,2)}{G_{2,4}^{1,2}(-|z|^2\,|\,0)}}{\left[m - \frac{G_{2,4}^{1,2}(-|z|^2\,|\,1)}{G_{2,4}^{1,2}(-|z|^2\,|\,0)}\right]^2}. \tag{56}$$

So, the statistical properties of the state $|z, k; m\rangle$ are dependent on the analytical properties of the above ratio of Meijer's G-functions (47), with $l = 0, 1, 2$. The analytic expression of these ratios is difficult to express, so the intensity correlation function (as well as the Mandel parameter) must be calculated numerically.

4. Statistical properties

We consider a quantum gas of the PHOs in thermodynamic equilibrium with the reservoir (the thermostat) at temperature T, which obeys the quantum canonical distribution. Also, we consider that the individual PHOs are in such states which are labelled by number state vectors $|v + m, k\rangle$, where m is a positive integer and indicates the number of added quanta. The corresponding normalized density operator for a fixed m and a fixed rotational quantum number J (or, equivalently, via equations (4) and (14), for a fixed number k) is then

$$\rho_J^{(m)} \equiv \rho_k^{(m)} = \frac{1}{Z_k^{(m)}} \sum_{v=0}^{\infty} \mathrm{e}^{-\beta E_{v+m,J}} |v + m, k\rangle\langle v + m, k| \tag{57}$$

where m is chosen to be a positive integer, $\beta = (k_B T)^{-1}$, k_B is Boltzmann's constant, and $Z_J^{(m)} = Z_k$ is the normalization constant, i.e. the partition function for a certain rotational state J.

The Q-function, i.e. the diagonal elements of the density operator in the representation of PA-BGCSs, is

$$\langle z, k; m|\rho_J^{(m)}|z, k; m\rangle \equiv \left\langle\rho_k^{(m)}\right\rangle_{z,k} = \frac{1}{Z_k^{(m)}} \sum_{v=0}^{\infty} \mathrm{e}^{-\beta E_{vJ}} |\langle z, k; m|v + m; k\rangle|^2 \tag{58}$$

which, using equations (21), (3) and (11), leads to

$$\left\langle\rho_k^{(m)}\right\rangle_{z,k} = \frac{1}{Z_k^{(m)}} \mathrm{e}^{-\beta E_{mJ}} [C_m(|z|)]^2 \sum_{v=0}^{\infty} \frac{\left(|z|^2 \frac{\overline{n}}{\overline{n}+1}\right)^v}{\rho_m(v, k)}. \tag{59}$$

Here we have used the well-known notation for the thermal mean occupancy for a HO with the angular frequency $\omega = 2\omega_0$

$$\overline{n} \equiv \frac{1}{\mathrm{e}^{2\beta\hbar\omega_0} - 1}. \tag{60}$$

The infinite series is of the same kind as that in equation (45), where $n = 0$ and $x = |z|^2 \exp(-2\beta\hbar\omega_0) = |z|^2 \frac{\overline{n}}{\overline{n}+1}$:

$$\begin{aligned}\left\langle\rho_k^{(m)}\right\rangle_{z,k} = &\frac{1}{Z_k^{(m)}} \mathrm{e}^{-\beta E_{mJ}} [C_m(|z|)]^2 G_{2,4}^{1,2} \\ &\times \left(-|z|^2 \frac{\overline{n}}{\overline{n}+1} \middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k \end{matrix}\right).\end{aligned} \tag{61}$$

By normalizing the density operator to unity, i.e.

$$\mathrm{Tr}\, \rho_k^{(m)} = \int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} W_k^{(m)}(|z|) \langle z, k; m|\rho_k^{(m)}|z, k; m\rangle = 1 \tag{62}$$

performing the angular integration, using the multiplication property of the Meijer's G-function and integrating the product of G-functions (see appendix A), we obtain the expression of the partition function

$$Z_k^{(m)} = \mathrm{e}^{-\beta(E_{mJ} - \hbar\omega_0)} \frac{1}{2\sinh\beta\hbar\omega_0} = \mathrm{e}^{-2\beta m\hbar\omega_0} Z_k^{(0)} = \left(\frac{\overline{n}}{\overline{n}+1}\right)^m Z_k^{(0)} \tag{63}$$

where

$$Z_k^{(0)} = \mathrm{e}^{\beta m_r \omega_0^2 r_0^2} \frac{\mathrm{e}^{-\beta\hbar\omega_0(2k-1)}}{2\sinh\beta\hbar\omega_0} \tag{64}$$

is the partition function for the usual (i.e. for $m = 0$) BGCSs of the PHO (see equation (70) of our previous paper [9]).

Consequently, the diagonal elements of the density matrix may be written as

$$\begin{aligned}\left\langle\rho_k^{(m)}\right\rangle_{z,k} = &2\,\mathrm{e}^{\beta(E_{mJ} - \hbar\omega_0)} \sinh\beta\hbar\omega_0 \\ &\times \frac{G_{2,4}^{1,2}\left(-|z|^2 \frac{\overline{n}}{\overline{n}+1} \middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k \end{matrix}\right)}{G_{2,4}^{1,2}\left(-|z|^2 \middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k \end{matrix}\right)}.\end{aligned} \tag{65}$$

Because the PA-BGCSs $|z, k; m\rangle$ form an overcomplete set of states, they may be used as a basis set despite the fact that they are non-orthogonal. Let us perform the diagonal expansion of the density operator $\rho_k^{(m)}$ in the PA-BGCSs (for the CS of HO this diagonal representation was introduced independently by Glauber [21] and Sudarshan [22] and is usually called the Glauber–Sudarshan representation):

$$\rho_k^{(m)} = \int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} W_k^{(m)}(|z|)|z, k; m\rangle P_k^{(m)}(z; \beta)\langle z, k; m|. \tag{66}$$

In [17] it is pointed out that the quasi-probability distribution function $P_k^{(m)}(z; \beta)$ can have negative values and strong singularities (it can become more singular than a delta function somewhere in the complex z-plane), especially when the density operator corresponds to a non-classical state with sub-Poissonian statistics. So, the diagonal P-representation of the density operator behaves well in describing non-classical states of light.

In order to find the function $P_k^{(m)}(z; \beta)$ let us begin with the diagonal elements of the density operator $\rho_k^{(m)}$ in the basis of the number states $|v + m; k\rangle$

$$\langle v+m; k|\rho_k^{(m)}|v+m; k\rangle = \int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} W_k^{(m)}(|z|)\langle v+m; k|z, k; m\rangle P_k^{(m)}(z; \beta)\langle z, k; m|v+m; k\rangle. \tag{67}$$

The diagonal elements are well known from equation (57)

$$\langle v+m; k|\rho_k^{(m)}|v+m; k\rangle = \frac{1}{Z_k^{(m)}} \mathrm{e}^{-\beta E_{v+m;J}} = 2\sinh\beta\hbar\omega_0\, \mathrm{e}^{-\beta\hbar\omega_0(2v+1)} \tag{68}$$

as well as the scalar product from equation (21)

$$\langle v+m; k|z, k; m\rangle = C_m(|z|)\frac{z^v}{\sqrt{\rho_m(v; k)}}. \tag{69}$$

After these suitable substitutions, using equation (42), performing the angular integration and the variable change $X = r^2 \exp(2\beta\hbar\omega_0) = r^2\frac{\overline{n}+1}{\overline{n}}$, we obtain

$$2\sinh\beta\hbar\omega_0 \left(\frac{\overline{n}+1}{\overline{n}}\right)^{m+1} \sqrt{\frac{\overline{n}}{\overline{n}+1}} = \frac{\Gamma(v+m+1)\Gamma(v+m+2k)}{[\Gamma(v+1)]^2[\Gamma(v+2k)]^2} \int_0^\infty \mathrm{d}X X^{v+m} \times G_{2,4}^{4,0}\left(X\frac{\overline{n}}{\overline{n}+1}\middle| \begin{array}{llll} 0, & 2k-1 & & \\ -m, & -m, & 2k-1-m, & 2k-1-m \end{array}\right) P_k^{(m)}(X; \beta). \tag{70}$$

Now, it is obvious that the P-function must have the following structure

$$P_k^{(m)}(X; \beta) = C^{(m)}(\beta) \frac{f_k^{(m)}(X)}{G_{2,4}^{4,0}\left(X\frac{\overline{n}}{\overline{n}+1}\middle| \begin{array}{llll} 0, & 2k-1 & & \\ -m, & -m, & 2k-1-m, & 2k-1-m \end{array}\right)} \tag{71}$$

and the radial integral is

$$\int_0^\infty \mathrm{d}X X^{v+m} f_k^{(m)}(X) = \frac{[\Gamma(v+1)]^2[\Gamma(v+2k)]^2}{\Gamma(v+m+1)\Gamma(v+m+2k)} \tag{72}$$

i.e. the same integral as equation (39). Then, the solution is

$$g_m(X) = G_{2,4}^{4,0}\left(X\middle| \begin{array}{llll} 0, & 2k-1 & & \\ -m, & -m, & 2k-1-m, & 2k-1-m \end{array}\right). \tag{73}$$

After the identification of the normalization constant $C^{(m)}(\beta)$, the final expression of the P-function is

$$P_k^{(m)}(X;\beta) = \frac{1}{\overline{n}}\left(\frac{\overline{n}+1}{\overline{n}}\right)^m \frac{G_{2,4}^{4,0}\left(|z|^2\frac{\overline{n}+1}{\overline{n}}\middle| \begin{matrix} 0, & 2k-1 & & \\ -m, & -m, & 2k-1-m, & 2k-1-m \end{matrix}\right)}{G_{2,4}^{4,0}\left(|z|^2\middle| \begin{matrix} 0, & 2k-1 & & \\ -m, & -m, & 2k-1-m, & 2k-1-m \end{matrix}\right)}. \tag{74}$$

It is not difficult to prove that the P-function satisfies the normalization condition

$$\int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} W_k^{(m)}(|z|) P_k^{(m)}(z;\beta) = 1. \tag{75}$$

In this way, the diagonal representation of the normalized density operator of the PHO in the PA-BGCS representation is

$$\begin{aligned}\rho_k^{(m)} = \frac{1}{\overline{n}}\int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} G_{2,4}^{1,2}\left(-|z|^2\middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k \end{matrix}\right)\\ \times G_{2,4}^{4,0}\left(|z|^2\frac{\overline{n}+1}{\overline{n}}\middle| \begin{matrix} m, & 2k-1+m & & \\ 0, & 0, & 2k-1, & 2k-1 \end{matrix}\right)|z,k;m\rangle\langle z,k;m|.\end{aligned} \tag{76}$$

Then the thermal expectation value (the thermal average) of an observable A concerning the PHO is given by

$$\begin{aligned}\langle A\rangle_k^{(m)} &= \mathrm{Tr}\left(\rho_k^{(m)} A\right)\\ &= \frac{1}{\overline{n}}\int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} G_{2,4}^{1,2}\left(-|z|^2\middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k \end{matrix}\right)\\ &\quad\times G_{2,4}^{4,0}\left(|z|^2\frac{\overline{n}+1}{\overline{n}}\middle| \begin{matrix} m, & 2k-1+m & & \\ 0, & 0, & 2k-1, & 2k-1 \end{matrix}\right)\langle A\rangle_{z,k}^{(m)}.\end{aligned} \tag{77}$$

In many cases (e.g. with the aim of calculating the thermal averages of N and N^2) it is necessary to solve the integrals involving products of two G-functions of the following kind

$$\begin{aligned}I(l) \equiv \int\int_{\mathcal{C}} \frac{\mathrm{d}^2 z}{\pi} G_{2,4}^{1,2}\left(-|z|^2\middle| \begin{matrix} -m, & 1-m-2k & & \\ 0, & l, & 1-2k, & 1-2k \end{matrix}\right)\\ \times G_{2,4}^{4,0}\left(|z|^2\frac{\overline{n}+1}{\overline{n}}\middle| \begin{matrix} m, & 2k-1+m & & \\ 0, & 0, & 2k-1, & 2k-1 \end{matrix}\right).\end{aligned} \tag{78}$$

When we perform the integration (see appendix A), we obtain

$$I(l) = G_{2,2}^{1,2}\left(-\frac{\overline{n}}{\overline{n}+1}\middle| \begin{matrix} 0, & 0 \\ 0, & l \end{matrix}\right) = (-1)^l l!\,(\overline{n})^{l+1}. \tag{79}$$

In the latter equation we have used the property (A.129) and the relation [14]

$$G_{1,1}^{1,1}\left(ax^\alpha\middle| \begin{matrix} \frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} \end{matrix}\right) = a^{\frac{\beta}{\alpha}}\frac{x^\beta}{1+ax^\alpha}. \tag{80}$$

By using this ansatz, the thermal expectation values of the number operator N and of the square of the number operator N^2 respectively are

$$\langle N\rangle_k^{(m)} = \frac{1}{\overline{n}}[mI(0) - I(1)] = (m + \overline{n}) \equiv \langle N\rangle^{(m)} \tag{81}$$

$$\langle N^2\rangle_k^{(m)} = \frac{1}{\overline{n}}[m^2 I(0) - (2m+1)I(1) + I(2)] = (m + \overline{n})^2 + \overline{n}(\overline{n}+1) \equiv \langle N^2\rangle^{(m)}. \tag{82}$$

When we put $m = 0$, we obtain the same expressions as the Bose–Einstein thermal distribution (see equations (83) and (84) of our previous paper [9]) and, consequently, we can say that the PHO is suitable to be associated with a boson (e.g. a photon).

On the other hand, we observe that these expectations are independent of the Bargmann index k, i.e. in the previous analysis they are independent of the rotational quantum number J.

With these expectations, we can define and calculate the thermal second-order correlation function $(g^{(2)})_k^{(m)}$ (i.e. the thermal analogue of the second-order correlation function for the state $|z, k; m\rangle$)

$$(g^{(2)})_k^{(m)} \equiv \frac{\langle N^2\rangle_k^{(m)} - \langle N\rangle_k^{(m)}}{\left(\langle N\rangle_k^{(m)}\right)^2} = 1 + \frac{(\overline{n})^2 - m}{(m + \overline{n})^2} \equiv (g^{(2)})^{(m)}. \tag{83}$$

The thermal analogue of the Mandel parameter is

$$Q_k^{(m)} = \langle N\rangle_k^{(m)} \left[(g^{(2)})^{(m)} - 1\right] = \frac{(\overline{n})^2 - m}{m + \overline{n}} \equiv Q^{(m)}. \tag{84}$$

These quantities are, also, independent of k.

Regarded as the function of the thermal mean occupancy $\overline{n}$, the thermal analogue of the Mandel parameter $Q^{(m)}$ takes the following extreme values

$$\lim_{\overline{n}\to 0} Q^{(m)}(\overline{n}) = -1 \qquad \lim_{\overline{n}\to\infty} Q^{(m)}(\overline{n}) = \infty \tag{85}$$

and also

$$\lim_{\overline{n}\to\sqrt{m}} Q^{(m)}(\overline{n}) = 0. \tag{86}$$

The latter value is the limit which determines the characteristics of sub-Poissonian, Poissonian and supra-Poissonian thermal states of the PHO. Thus, the condition for the existence of sub-Poissonian statistics is

$$\overline{n} < \sqrt{m} \tag{87}$$

implying that the field is non-classical.

In the particular case of the usual BGCSs, i.e. for $m = 0$, we obtain

$$(g^{(2)})^{(0)} = 2 \qquad Q^{(0)} = \overline{n}. \tag{88}$$

This equation points out a new significance of the thermal analogue of the Mandel parameter; for the usual BGCSs it represent the thermal mean occupancy.

The normalized density operator characterizes the quantum gas of PHOs, in the states with m added photons, regarded as the whole quantum system

$$\rho^{(m)} = \frac{1}{Z^{(m)}} \sum_J (2J+1) Z_J^{(m)} \rho_J^{(m)} \tag{89}$$

where $\rho_J^{(m)} \equiv \rho_k^{(m)}$ and $Z^{(m)}$ are the diagonal representation of the normalized density operator and the total partition function for the rotational state J and m added photons, respectively (see equation (57)).

Consequently, the total thermal expectation value of an observable A is

$$\langle A\rangle^{(m)} = \mathrm{Tr}\, A\rho^{(m)} = \frac{1}{Z^{(m)}}\sum_J (2J+1) Z_J^{(m)}\, \mathrm{Tr}\, A\rho_J^{(m)} \tag{90}$$

where $\mathrm{Tr}\, A\rho_J = \langle A\rangle_J^{(m)} = \langle A\rangle_k^{(m)}$ is the thermal expectation value for the rotational state J and m added photons (see equation (77)).

Similarly, the total partition function is

$$Z^{(m)} = \sum_J (2J+1)\sum_v \mathrm{e}^{-\beta E_{v+m,J}} = \sum_J (2J+1) Z_J^{(m)} = \sum_J (2J+1)\left(\frac{\overline{n}}{\overline{n}+1}\right)^m Z_k^{(0)} \tag{91}$$

where $Z_J^{(m)} \equiv Z_k^{(m)}$ is the vibrational partition function for the states with fixed rotational number J (or fixed k) and m added photons. Similarly, $Z_J^{(0)} \equiv Z_k^{(m)}$ is the same for $m = 0$.

The expression for this last quantity is

$$Z_k^{(m)} = \mathrm{e}^{\beta m_r \omega_0^2 r_0^2}\frac{1}{2\sinh \beta\hbar\omega_0}\,\mathrm{e}^{-x\alpha} \tag{92}$$

where we have denoted $x = \beta\hbar\omega_0$.

Also, if we use the notation

$$T_\alpha(x) = \sum_{J=0}^{\infty} (2J+1)\,\mathrm{e}^{-x\alpha} \tag{93}$$

the total partition function becomes

$$Z^{(m)} = \left(\frac{\overline{n}}{\overline{n}+1}\right)^m Z^{(0)}. \tag{94}$$

The corresponding expression for $m = 0$ was obtained in paper PI (see equation (89) of [9])

$$Z^{(0)} = \mathrm{e}^{\beta m_r \omega_0^2 r_0^2}\frac{1}{2\sinh \beta\hbar\omega_0} T_\alpha(x). \tag{95}$$

We see that the contribution of m added photons in the partition function is separated, which has as a consequence the possibility to evince the corresponding contribution in the expressions of internal and free energies, as seen in the following.

In paper PI, we have analysed the conditions for which all these expressions lead, at the harmonic limit, to the corresponding results for the HO-3D [9].

The free energy per particle (per PHO) is

$$F^{(m)} = -\frac{1}{\beta}\ln Z^{(m)} = m\frac{1}{\beta}\ln\left(\frac{\overline{n}+1}{\overline{n}}\right) + F^{(0)} \tag{96}$$

where $F^{(0)}$ is the free energy for the PHO quantum gas without added photons

$$F^{(0)} = -\frac{1}{\beta}\ln Z^{(0)} = -m_r\omega_0^2 r_0^2 + \frac{1}{\beta}\ln 2\sinh\beta\hbar\omega_0 - \frac{1}{\beta}\ln T_\alpha. \tag{97}$$

Similarly, the internal energy per particle can be calculated in two ways: (a) directly, by using the partition function

$$U^{(m)} = -\frac{\partial}{\partial\beta}\ln Z^{(m)} \tag{98}$$

and (b) by using the general equation (77) for the thermal expectations

$$U^{(m)} = \langle H_\alpha\rangle^{(m)} = \frac{1}{Z^{(m)}}\sum_J (2J+1) Z_k^{(m)}\langle H_\alpha\rangle_k^{(m)} \tag{99}$$

where the expectation of the Hamiltonian H_α is calculated using equations (6) and (5).

Both ways lead to the following result

$$U^{(m)} = 2m\hbar\omega_0 + U^{(0)} \tag{100}$$

where $U^{(0)}$ is the internal energy for the PHO quantum gas without added photons which was calculated in paper PI (see equation (95)) [9]

$$U^{(0)} = -m_r\omega_0^2 r_0^2 + \hbar\omega_0\left[\coth\beta\hbar\omega_0 - \frac{\partial}{\partial x}\ln T_\alpha\right]. \tag{101}$$

Then, for the entropy of the PHO

$$S^{(m)} = k_B\beta(U^{(m)} - F^{(m)}) \tag{102}$$

we obtain the following expression

$$S^{(m)} = mk_B\left(\beta 2\hbar\omega_0 + \ln\frac{\overline{n}}{\overline{n}+1}\right) + S^{(0)} \tag{103}$$

where

$$S^{(0)} = k_B\beta(U^{(0)} - F^{(0)}) \tag{104}$$

is the entropy of the PHO quantum gas without added photons which was calculated in paper PI (equation (97))

$$S^{(0)} = k_B(\beta\hbar\omega_0\coth\beta\hbar\omega_0 - \ln 2\sinh\beta\hbar\omega_0) + k_B\left(\ln T_\alpha - \beta\hbar\omega_0\frac{\partial}{\partial x}\ln T_\alpha\right). \tag{105}$$

In the above expressions for the free energy, the internal energy and the entropy we have evinced the contribution of added photons.

Finally, the molar heat capacity at the constant volume

$$C_V^{(m)} = \frac{1}{\nu}\frac{\partial U^{(m)}}{\partial T} = -\frac{1}{\nu}k_B\beta^2\frac{\partial U^{(m)}}{\partial\beta} \tag{106}$$

is not influenced by the presence of added photons (see equation (99) of our previous paper [9])

$$\frac{C_V^{(m)}}{R} = \left[\left(\frac{x}{\sinh x}\right)^2 + x^2\frac{\partial^2}{\partial x^2}(\ln T_\alpha)\right] = \frac{C_V^{(0)}}{R} \tag{107}$$

where R is the constant of the ideal gas.

5. The time dependence

At the end of this paper, we refer to the time dependence of the PA-BGCSs. The time-dependent BGCSs of the PHO $|z, k; t\rangle \equiv |z, k; 0; t\rangle$ was previously obtained by acting on the time-independent BGCS $|z, k\rangle \equiv |z, k; 0; 0\rangle$ by the evolution operator $U(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}H_k t\right)$ [9]

$$|z, k; 0; t\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_k t}|z, k; 0; 0\rangle. \tag{108}$$

Also, the unitary evolution operator $U(t)$ transforms the PA-BGCS into

$$\begin{aligned}|z, k; m; t\rangle &= U(t)|z, k; m; t\rangle \equiv U(t)|z, k; m\rangle \\ &= N_m(|z|)[U(t)(K_+)^m U^+(t)]U(t)|z, k\rangle \equiv N_m(|z|)[K_+(t))]^m|z, k; 0; t\rangle\end{aligned} \tag{109}$$

where the time-dependent raising operator is

$$K_+(t) = U(t)K_+U^+(t) \tag{110}$$

and satisfies the initial condition $K_+(0) = K_+$.

The time-dependent lowering operator is defined similarly

$$K_-(t) = U(t)K_-U^+(t). \tag{111}$$

By rewriting equation (15) as follows

$$K_-|z, k; 0; 0\rangle = z|z, k; 0; 0\rangle \tag{112}$$

and using the unitarity of the evolution operator, we obtain

$$K_-(t)|z, k; 0; t\rangle = z|z, k; 0; t\rangle. \tag{113}$$

On the other hand, the explicit time dependence of the PA-BGCSs can be obtained by expressing the result of the action of the evolution operator on the state $|z, k; m; 0\rangle$, using equation (21).

The radial Schrödinger equation for the PHO is

$$H_k|v + m, k\rangle = \left[\hbar\omega_0(2v + 2m + 2k) - m_r\omega_0^2 r_0^2\right]|v + m, k\rangle = (E_{m,J} + \hbar\omega_0 2v)|v + m, k\rangle \tag{114}$$

where, according to equations (5), (6) and (14), we have

$$H_k \equiv H_\alpha = \hbar\omega_0 H_\alpha^{(\mathrm{red})}(y) - m_r\omega_0^2 r_0^2. \tag{115}$$

After the calculations we obtain

$$|z, k; m; t\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_{m,J}t} C_m(|z|) \sum_{v=0}^{\infty} \frac{\left(z\,\mathrm{e}^{-\mathrm{i}2\omega_0 t}\right)^v}{\sqrt{\rho_m(v; k)}}|v + m, k\rangle. \tag{116}$$

When we use the notation

$$z(t) = z\,\mathrm{e}^{-2\mathrm{i}\omega_0 t} \tag{117}$$

we obtain

$$|z(t), k; m\rangle = C_m(|z(t)|) \sum_{v=0}^{\infty} \frac{[z(t)]^v}{\sqrt{\rho_m(v; k)}}|v + m, k\rangle. \tag{118}$$

In this way, we obtain the following time dependence of the PA-BGCSs

$$|z, k; m; t\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_{m,J}t}|z(t), k; m\rangle. \tag{119}$$

The time-dependent photon-added coherent states have applications when the Hamiltonian is time dependent. As pointed out in [12] (and references therein) relating to the PACS for the usual HO, we believe also that the PA-BGCSs may have applications when the PHO (the field mode) angular frequency is time variable, according to a known function $\omega(t)$ (also, for a HO with a singular perturbation [26–28]). In this case, the initial PA-BGCSs $|z, k; m; 0\rangle$, created in some way in a cavity or in a ion trap, evolves at $t > 0$ according to equation (119). The time-dependent coherent states of the generalized time-dependent parameter oscillator [29] will be useful for future studies in quantum optics as well as in atomic and molecular physics.

6. Concluding remarks

In this paper, we have constructed and studied the properties of an interesting class of non-classical states, namely the PA-BGCSs of the PHO. These states can be obtained by repeated application of the raising operator K_+ on the basis coherent states $|z, k\rangle$, i.e. on the usual BGCSs which, for the PHO, were deduced and studied previously [9].

Physical properties of closely-related states to those discussed in the present paper have been recently studied in [35]. These states are squeezed number states obtained through the use of the group operator, rather than as eigenstates of the algebra operator.

The PA-BGCSs $|z, k; m\rangle$ are intermediate states between the usual BGCSs $|z, k\rangle$ and the number states $|m; k\rangle$ (the eigenstates of the number operator N)

$$|z, k; m\rangle \longrightarrow \begin{cases} |z, k\rangle & \text{if} \quad m \to 0, \qquad z, k = \text{const} \\ |m; k\rangle & \text{if} \quad m, k = \text{const} \qquad z \to 0. \end{cases} \tag{120}$$

The properties and some applications of such states, intermediate or interpolating between the coherent and the number states, have been recently investigated well in [36].

We have developed the PA-BGCSs in terms of the number states $|v; k\rangle$ and we find the analytical form of the positive weight function $W_k^{(m)}(|z|)$ in the corresponding resolution of the unity. For fixed z and k (the Bargmann index), PA-BGCSs are normalized for any finite m. As pointed out in [30], due to the structure of the function $\rho_m(v; k)$ (equation (20)), the normalization constant $C_m(|z|)$ (27), the weight function $W_k^{(m)}(|z|)$ and, consequently, the expectation values are always expressible through certain generalized hypergeometrical series ${}_pF_q(\ldots; x)$ which are special cases of the general Meijer's G-function. In this way, PA-BGCSs have become an interesting application field of the general Meijer's G-function.

In section 3 we have calculated some expectation values related to the first two powers of the number operator N in order to evaluate the characteristic of the field, i.e. we have calculated the second-order correlation function $(g^2)_{z,k}^{(m)}$ and the Mandel parameter $Q_{z,k}^{(m)}$. As is known, the Mandel parameter is a convenient measure of the deviation of the photon number statistics from the Poisson distribution [31]. It vanishes for the Poisson distribution, and is positive or negative according to whether the distribution is super-Poissonian (bunching effect) or sub-Poissonian (antibunching effect).

In section 4 of this paper we have examined the statistical properties of the quantum gas of the PHOs in thermodynamical equilibrium with the reservoir at temperature T, which obeys the quantum canonical distribution. We have constructed the density matrix in the PA-BGCSs representation and, especially, its diagonal representation and we have found the P-function. In order to prove the correctness of the expressions we have obtained for the density operator $\rho^{(m)}$, we have calculated some thermal expectation values (thermal averages) for the few observables under consideration for the PHO quantum gas (the thermal analogue of the second correlation function) $(g^2)^{(m)}$ and the Mandel parameter $Q^{(m)}$, as well as the free and internal energies, the entropy and molar heat capacity at the constant volume). In the expressions for the free and internal energies and the entropy, we have evinced the contribution of the added photons, while the added photons do not contribute to the molar heat capacity at constant volume.

In our opinion, the whole construction of the PA-BGCSs for the PHO seems to be a new result because, to our knowledge, this approach has not yet appeared in the literature. Among these new obtained expressions, the main results seem to be the expressions for the weight function $W_k^{(m)}(|z|)$ and for the P-function.

The same as for other families of coherent states, a legitimate open question is how to construct photon-substracted BGCSs for the PHO, i.e. the states obtained by acting with the lowering operator K_- on some state $|\widetilde{z}, k\rangle$ in order to produce the normalized state $|\widetilde{z}, k; m\rangle = \widetilde{N}_m(|\widetilde{z}|)(K_-)^m|\widetilde{z}, k\rangle$. For the usual HO this problem has recently been studied in detail in [23–25]. Contemporaneously with Barut and Girardello, many authors have studied and applied these coherent states (e.g. $SU(1, 1)$ coherent states applied to superfluidity in [34]).

Because $|z, k\rangle$ is an eigenstate of $(K_-)^m$ (see equation (15) and [9]), it follows that the state $|\widetilde{z}, k\rangle$ must be different from the state $|z, k\rangle$ and, as a consequence, the photon-subtracted BGCS for the PHO would not be normalizable at $z = 0$.

Here we point out that, at the limit $m \to 0$, we have

$$\lim_{m\to 0} \mathcal{F}^{(m)} = \mathcal{F}^{(0)} \tag{121}$$

i.e. each formula or expression which is related to PA-BGCS ($\mathcal{F}^{(m)}$) leads, at the limit $m \to 0$, to the corresponding formula or expression related to BGCS ($\mathcal{F}^{(0)}$). In this way, we recover all the results obtained for the usual BGCSs in our previous paper [9].

Finally, we observe that the HO-3D can be considered as a limit oscillator of the PHO. This limit is called the harmonic limit of the PHO and, for a certain physical observable A, is defined as (see [32])

$$\lim_{\substack{\omega \to 2\omega_0 \\ r_0 \to 0 \\ \alpha \to J+\frac{1}{2}}} \mathcal{F}^{(\mathrm{PHO})} \equiv \lim_{\mathrm{HO}} \mathcal{F}^{(\mathrm{PHO})} = \mathcal{F}^{(\mathrm{HO})}. \tag{122}$$

Here the quantities with the superscript (PHO) correspond to the PHO with the angular frequency ω, while the same quantities with the superscript (HO) correspond to the HO-3D (with the frequency ω_0).

This assertion may be a supplementary argument for the investigation of the behaviour and properties of the PHO.

Acknowledgment

The author wishes to thank Professor Dr Karol A Penson from Université Pierre et Marie Curie of Paris for communicating the results prior to publication.

Appendix A

The generalized hypergeometric series are defined as [14]

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{v=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_v}{\prod_{j=1}^{q} (b_j)_v} \frac{x^v}{v!} \tag{A.123}$$

with $p \leqslant q$ and $|z| < 1$, while

$$(a_j)_v = \frac{\Gamma(a_j + v)}{\Gamma(a_j)}. \tag{A.124}$$

The generalized hypergeometric function is related to Meijer's G-function as follows [14]

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} G^{1,p}_{p,q+1}\left(-x \middle| \begin{matrix} (1-a_p) & \\ 0, & (1-b_q) \end{matrix}\right). \tag{A.125}$$

The multiplication formula of Meijer's G-function [14]

$$x^{\sigma} G^{m,n}_{p,q}\left(x \middle| \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right) = G^{m,n}_{p,q}\left(x \middle| \begin{matrix} (a_p + \sigma) \\ (b_q + \sigma) \end{matrix}\right) \tag{A.126}$$

and the differential property [14]

$$x^l \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l G^{m,n}_{p,q}\left(x \middle| \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right) = (-1)^l G^{m,n+1}_{p+1,q+1}\left(x \middle| \begin{matrix} 0, & (a_p) \\ (b_q), & l \end{matrix}\right) \tag{A.127}$$

lead to the following integral involving products of G-functions [14, 15]

$$\int_0^\infty \mathrm{d}x\, G^{m,n}_{p,q}\left(\eta x \middle| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right) G^{\mu,\nu}_{\sigma,\tau}\left(\omega x \middle| \begin{matrix}(c_\sigma)\\(d_\tau)\end{matrix}\right)$$
$$= \frac{1}{\eta} G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}\left(\frac{\omega}{\eta} \middle| \begin{matrix}-(b_m), & (c_\sigma), & -b_{m+1}, & \dots, & -b_q\\ -(a_n), & (d_\tau) & -a_{n+1}, & \dots, & -a_p\end{matrix}\right). \tag{A.128}$$

If one of $j = m+1, \dots, q$ of b_j is equal to one of $k = 1, 2, \dots, n$ of a_k, the G-function reduces to one of lower order and the parameters p, q and n are each decreased by unity. Thus, if $p, q, n \geqslant 1$

$$G^{m,n}_{p,q}\left(x \middle| \begin{matrix}a_1, & \dots, & a_p & \\ b_1, & \dots, & b_{q-1}, & a_1\end{matrix}\right) = G^{m,n-1}_{p-1,q-1}\left(x \middle| \begin{matrix}a_2, & \dots, & a_p\\ b_1, & \dots, & b_{q-1}\end{matrix}\right). \tag{A.129}$$

Appendix B

We consider the general sum

$$S_n^{(m)} \equiv \sum_{v=0}^{\infty} \frac{x^v}{\rho_m(v,k)} v^n \tag{B.130}$$

from which, for the particular case $n = 0$, we obtain the fundamental sum

$$S_0^{(m)} = \sum_{v=0}^{\infty} \frac{x^v}{\rho_m(v,k)} = \frac{\Gamma(m+1)\Gamma(m+2k)}{[\Gamma(2k)]^2} {}_2F_3(m+1, m+2k; 1, 2k, 2k; x). \tag{B.131}$$

On the other hand, due to the relation between the hypergeometric series and Meijer's G-function (A.125), using equation (27), we obtain

$$S_0^{(m)} = G^{1,2}_{2,4}\left(-x \middle| \begin{matrix}-m, & 1-m-2k & & \\ 0, & 0, & 1-2k, & 1-2k\end{matrix}\right) = [C_m(\sqrt{x})]^{-\frac{1}{2}}. \tag{B.132}$$

Moreover, each power v^n can be written as a linear combination of the binomial products

$$v^n = \sum_{l=1}^{n} c_l^{(n)} \frac{v!}{(v-l)!} = \sum_{l=1}^{n} c_l^{(n)} \binom{v}{l} l! \tag{B.133}$$

where the coefficients $c_l^{(n)}$ which depend on n can easily be determined.

The above equality is useful to express the sum $S_n^{(m)}$ with $n > 0$ through the derivatives of the fundamental sum $S_0^{(m)}$ as follows

$$S_n^{(m)} = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n S_0^{(m)} = \sum_{l=1}^{n} c_l^{(n)} x^l \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l S_0^{(m)}. \tag{B.134}$$

By using the property (B.133) and then applying the properties (A.127) and (A.129) to the equation (B.132), finally we obtain

$$S_n^{(m)} = \sum_{l=1}^{n} (-1)^l c_l^{(n)} G^{1,2}_{2,4}\left(-x \middle| \begin{matrix}-m, & 1-m-2k & & \\ 0, & l, & 1-2k, & 1-2k\end{matrix}\right)$$
$$\equiv \sum_{l=1}^{n} (-1)^l c_l^{(n)} G^{1,2}_{2,4}(-x|l). \tag{B.135}$$

References

[1] Gol'dman I I, Krivchenkov V D, Kogan V I and Galitskii V M 1960 *Problems in Quantum Mechanics* (London: Infosearch) p 8
[2] Sage M 1984 *Chem. Phys.* **87** 431–9
[3] Sage M and Goodisman M J 1985 *Am. J. Phys.* **53** 350–5
[4] Ballhausen C J 1988 *Chem. Phys. Lett.* **146** 449
Ballhausen C J 1988 *Chem. Phys. Lett.* **151** 428
Ballhausen C J 1989 *Chem. Phys. Lett.* **154** 174
[5] Büyükkilic F, Demirhan D and Özeren S F 1992 *Chem. Phys. Lett.* **194** 9–12
[6] Popov D 1995 *Acta Phys. Slovaca* **45** 557–66
[7] Büyükkilic F, Egrifes H and Demirhan D 1997 *Theor. Chem. Accounts (Theor. Chim. Acta)* **98** 192–6
[8] Popov D 1998 *Acta Phys. Slovaca* **48** 1–12
[9] Popov D 2001 *J. Phys. A: Math. Gen.* **34** 5283
[10] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[11] Agarwal G S and Tara K 1991 *Phys. Rev.* A **43** 492
Agarwal G S and Tara K 1992 *Phys. Rev.* A **46** 485
[12] Dodonov V V, Marchiolli M A, Korennoy Ya A, Man'ko V I and Mouchin Y A 1998 *Phys. Rev.* A **58** 4087
[13] Sixderniers J-M and Penson K A 2001 *J. Phys. A: Math. Gen.* **34** 2859
[14] Mathai A M and Saxena R K 1973 *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences (Lecture Notes in Mathematics vol 348)* (Berlin: Springer)
[15] Erdelyi A and Bateman H 1973 *Higher Transcendental Functions* vol I (Moscow) (in Russian)
[16] Sixderniers J-M, Penson K A and Solomon A I 1999 *J. Phys. A: Math. Gen.* **32** 7543
[17] Brif C and Ben-Aryeh Y 1994 *Quantum Opt.* **6** 391
[18] Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer)
[19] Solomon A I 1994 *Phys. Lett.* A **196** 29
[20] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[21] Glauber R J 1963 *Phys. Rev.* **131** 2766
[22] Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277
[23] Dakna M, Anhut T, Opatrný T, Knöll L and Welsch D-G 1997 *Phys. Rev.* A **55** 3184
[24] Dakna M, Knöll L and Welsch D-G 1998 *Opt. Commun.* **145** 309
[25] Dakna M, Knöll L and Welsch D-G 1998 *Preprint* quant-ph/9803077
[26] Camiz P, Gerardi A, Marchioro C, Presutti E and Scaratelli E 1971 *J. Math. Phys.* **12** 2040
[27] Dodonov V V, Malkin I A and Man'ko V I 1972 *Phys. Lett.* A **39** 377
[28] Dodonov V V, Malkin I A and Man'ko V I 1974 *Nuovo Cimento* B **24** 46
[29] Lo C F 1993 *Europhys. Lett.* **24** 319
[30] Sixderniers J-M, Penson K A and Klauder J R 2001 *Int. J. Mod. Phys.* B **15** 4231
[31] Quesne C 2000 *Phys. Lett.* A **272** 313
[32] Popov D 1995 *Acta Phys. Slovaca* **45** 557
[33] Ali S T, Antoine J-P, Gazeau J-P and Mueller U A 1995 *Rev. Math. Phys.* **7** 1013
[34] Solomon A I 1971 *J. Math. Phys.* **12** 390
[35] Feng Y and Solomon A I 1998 *Opt. Commun.* **152** 299
[36] Fu H, Feng Y and Solomon A I 2000 *J. Phys. A: Math. Gen.* **33** 2231